ISSN: 2456-1363
International Journal of Scientific Research & Growth
A Multidisciplinary Peer Reviewed and Refereed International Journal
Volume-1 Issue-4 October 2016
www.ijsrg.com
editor@ijsrg.com

The Gereralized Mittag-Leffler Function And The Growth Model Of Population

[1]Kishan Sharma, [2]Sunil Pathak, & [3]Priyanka Swankar
[1]Amity University Madhya Pradesh, Gwalior M .P. India.
[2]P.G.V. College, Gwalior M. P. India.
Email- drkishansharma2006@rediffmail.com

Abstract

In the present paper, the authors calculated the size of population in terms of the generalized Mittag-Leffler function [5]. Some special cases are also discussed.

Mathematics Subject Classification: 26A33, 33C60, 44A15

Keywords and Phrases: *Fractional calculus operators generalized Mittag-Leffler function, exponential function.*

1- INTRODUCTION

The importance of Mittag-Leffler functions in physics is steadily increasing. It is simply said that deviations of physical phenomena from exponential behavior could be governed by physical laws through Mittag-Lefflerfunctions(power law). Currently more and more such phenomena are discovered and studied. It is particularly important for the disciplines of stochastic systems, dynamical systems theory and disordered systems.Eventually, it is believed that all these new research results will lead to the discovery of truly non-equilibrium statistical machanics. This is statistical mechanics beyond Boltzmann and Gibbs. This non-equilibrium statistical mechanics will focus on entropy production, reaction,diffusion, reaction-diffusion,, and so forth, and maybe governed by fractional calculus. Right now, fractional calculus and generalization of Mittag-Leffler functions are very important in research in physics.

The Mittag-Leffler function [3]

$$E_\alpha(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha r + 1)}, \quad (\alpha > 0) \qquad (1.1)$$

The generalized form of (1.1) given [5] as

$$E_{\alpha,\beta}(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha r + \beta)}, \quad (\alpha > 0) \qquad (1.2)$$

Under simplified conditions such as a constant environment (and with no migration), it can be shown that change in population size (x) through time (t) will depend on the difference between individual birth rate (b_0) and death rate (d_0), and given by:

$$\frac{dx}{dt} = (b_0 - d_0)x \qquad (1.3)$$

Where b_0=instantaneous birth rate, births per individual per time period (t)

d_0=instantaneous death rate, deaths per individual per time period (t), and

x=current population size.

The difference between birth and death rates (b_0-d_0) is called a, the intrinsic rate of natural increase, or the Malthusian parameter. It is the theoretical maximum number of individuals added to the population per individual per time. By solving the differential equation, we get a formula to estimate a population size at any time.

$$x=x_0e^{at} \tag{1.4}$$

The behavior of the exponentially growing function and its implications for the population growth of humans made for shocking and sensational controversy two hundred years ago, when it was described by Thomas R. Malthus in his 1798 publication. Malthus predicted that unless disasters or plagues limit the population on earth, simple exponential growth would result in unlimited population density whereas food supply could at best increase linearly. He concluded that mass-starvation, strife, and wars would be the lot of mankind.

The Riemann-Liouville operators of fractional calculus are defined in the books written by Miller and Ross [6] as

$$ {}_aD_t^{-\nu} N(t) = \frac{1}{\Gamma(\nu)}\int_a^t (t-u)^{\nu-1}N(u)du, \mathrm{Re}(\nu)>0, t>a \tag{1.5}$$

2- MAIN RESULT

In this section we will prove the following result presented in the form of theorem given below:

Theorem: For $\mathrm{Re}(\alpha)>0$, the population of some living species at time t is given by x (t) = $x_0\, E_{\alpha,1}(a^\alpha t^\alpha)$ (2.1)

Proof: Let $x(t)$ be the population of some living species at time t; also, let b and d be the intrinsic or specific birth and death rates, i.e. b and d are the number of births and deaths per individual per unit time. This gives us the equation due to Malthus [7].

$$\frac{dx}{dt}=ax \tag{2.2}$$

On integrating, we get

$$x(t)-x(0)=a\ {}_0D_t^{-1}x(t) \tag{2.3}$$

Where ${}_0D_t^{-1}$ is the standard Riemann-integral? On changing into the fractional integral form by replacing $a\ {}_0D_t^{-1}x(t)$ by $a^{-\alpha}\ {}_0D_t^{-\alpha}x(t)$, we get

$$x(t)-x(0)=a^{-\alpha}\ {}_0D_t^{-\alpha}x(t) \tag{2.4}$$

With initial condition x (t=0) = x_0

Using the Laplace transform, we get

$$L\{{}_0D_t^{-\alpha} f(t);s\}=s^{-\alpha}F(s), \tag{2.5}$$

Where F(s) $=\Gamma(s)\int_0^\infty e^{-su} f(u)\, du$,

Using (2.5) in (2.4), we get

$$x(s)-\frac{x_0}{s}=a^\alpha s^{-\alpha}x(s) \tag{2.6}$$

We arrive at

$$x(s)=x_0s^{-1}[1-a^\alpha s^{-\alpha}]^{-1} \tag{2.7}$$

By using binominal expansion, we get

$$x(s)= x_0s^{-1}\sum_{r=0}^{\infty}(a^\alpha s^{-\alpha})^r \tag{2.8}$$

Taking inverse Laplace transform, we obtain

$$x(t)=x_0\sum_{r=0}^{\infty} a^{\alpha r}\frac{t^{\alpha r}}{\Gamma(\alpha r+1)} \tag{2.9}$$

Using (1.1), we arrive at

$$x(\mathrm{t}) = x_0\, E_{\alpha,1}(a^{\alpha} t^{\alpha}) \qquad (2.10)$$

Where $E_{\alpha,1}(.)$ is the Mittag-Leffler function given by (1.2).

Special Case:

If we take $\alpha = 1$ in (2.1), we get the following relation [7]

$$x(\mathrm{t}) = x_0\, e^{a\,t} \qquad (3.1)$$

Where $e^{a\,t}$ is the exponential function [1].

REFERENCES

[1] E.D. Rainville, Special Functions, Chelsea Publishing Company, Bronx, New York (1960).

[2] J.N. Kapoor, Mathematical Models in Biology and Medicine, East-West Press Private Limited, New Delhi (1985).

[3] G.M. Mittag-Leffler, Sur la nouvelle function $E_{\alpha}(x)$, C. R. Acad. Sci. Paris, (137) (2) (1903), 554-558.

[4].G.M. Mittag-Leffler, Sur la representation analytique de'une branche uniforme une function monogene, *Acta. Math*. 29(1905), 101-181.

[5] A. Wiman, Uber die nullsteliun der fuctionen $E_{\alpha}(x)$, Acta Math., 29(1905), 217-234.

[6] K.S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, New York(1993).

[7] T. R. Malthus, An essay on the principal of population, John Murry. London (1826).

[8] K.Sharma et al, A note on the growth model of population and Mittag-Leffler function, Appl. Sci. Periodical 17(4)(2015),149.